

THE COMPUTATION OF GENERALIZED EHRHART SERIES IN NORMALIZ

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Let $M \subset \mathbb{Z}^n$ be an affine monoid endowed with a positive \mathbb{Z} -grading \deg . then the *Ehrhart* or *Hilbert series* is the generating function

$$E_M(t) = \sum_{x \in M} t^{\deg x} = \sum_{k=0}^{\infty} \#\{x \in M : \deg x = k\} t^k,$$

and $E(M, k) = \#\{x \in M : \deg x = k\}$ is the Ehrhart or Hilbert function of M (see [4] for terminology and basic theory). It is a classical theorem that $E_M(t)$ is the power series expansion of a rational function of negative degree at $t_0 = 0$ and that $E(M, k)$ is given by a quasipolynomial of degree $\text{rank } M - 1$ with constant leading coefficient equal to the (suitably normed) volume of the rational polytope

$$P = \text{cone}(M) \cap A_1$$

where $\text{cone}(M) \subset \mathbb{R}^n$ is the cone generated by M and A_1 is the hyperplane of degree 1 points. In the following we assume that

$$M = \text{cone}(M) \cap L$$

for a sublattice L of \mathbb{Z}^n . Then $E(M, k)$ counts the L -points in the multiple kP , and is therefore the Ehrhart function of P (with respect to L).

Monoids of the type just introduced are important for applications, and in some of them, like those discussed in Section 3, one is naturally led to consider *generalized* (or weighted) *Ehrhart series*

$$E_{M,f}(t) = \sum_{x \in M} f(x) t^{\deg x}$$

where f is a polynomial in n indeterminates. It is well-known that also the generalized Ehrhart series is the power series expansion of a rational function; see [1], [2].

In this note we describe the computation of generalized Ehrhart series based on Stanley decompositions [13]. Apart from taking the existence of Stanley decompositions as granted [13], we give complete and very elementary proofs of the basic facts. They follow exactly the implementation in the offspring `NmzIntegrate`¹ of `Normaliz`. The input polynomials of `NmzIntegrate` must have rational coefficients, and we assume that f is of this type although it is mathematically irrelevant.

The generalized Ehrhart function is given by a quasipolynomial of degree $\leq \deg f + \text{rank } M - 1$, and the coefficient of $k^{\deg f + \text{rank } M - 1}$ can easily be described as the integral of f over the polytope P . Therefore we have also included (and implemented) an approach

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¹`Nmzintegrate` will be uploaded to [6] together with `Normaliz 2.9` by February 2013.

to the computation of integrals of polynomials over rational polytopes in the spirit of the Ehrhart series computation. See [2] and [8] for more sophisticated approaches.

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1. THE COMPUTATION OF GENERALIZED EHRHART SERIES

Via a Stanley decomposition and substitution the computation of generalized Ehrhart series can be reduced to the case in which M is a free monoid, and for free monoids one can split off the variables of f successively so that one ends at the classical case $M = \mathbb{Z}_+$. We go the opposite direction, starting from \mathbb{Z}_+ .

1.1. The monoid \mathbb{Z}_+ . Let $M = \mathbb{Z}_+$. By linearity it is enough to consider the polynomials $f(k) = k^m$, $k \in \mathbb{Z}_+$, for which the generalized Ehrhart series is given by

$$\sum_{k=0}^{\infty} k^m t^{um}, \quad u = \deg 1,$$

and if necessary we can assume $u = 1$, substituting $t \mapsto t^u$ in the final result.

The rising factorials

$$(k+1)_m = (k+1) \cdots (k+m)$$

form a \mathbb{Z} -basis of the polynomial ring $\mathbb{Z}[k]$. Therefore we can write

$$(1.1) \quad k^m = \sum_{j=0}^m s_{m,j} (k+1)_j.$$

and use that

$$(1.2) \quad \sum_{k=0}^{\infty} (k+1)_r t^k = \left(\frac{1}{1-t} \right)^{(r)} = \frac{r!}{(1-t)^{r+1}}.$$

Equations (1.1) and (1.2) solve our problem for $M = \mathbb{Z}_+$ and $f(k) = k^m$:

$$(1.3) \quad \sum_{k=0}^{\infty} k^m t^{uk} = \frac{A_{m,u}(t)}{(1-t^u)^{m+1}}, \quad A_{m,u}(t) \in \mathbb{Z}[t].$$

It is enough to compute $A_{m,1}(t)$ because $A_{m,u}(t) = A_{m,1}(t^u)$. One should note that $A_{m,u}$ is a polynomial of degree m . Therefore the rational function in (1.3) has negative degree.

Since the coefficient $s_{m,m}$ of $(k+1)_m$ in the representation of k^m is evidently equal to 1, we have

$$(1.4) \quad \sum_{k=0}^{\infty} k^m t^{um} = \frac{m!}{(1-t)^{m+1}} + \text{terms of smaller pole order at } t = 1$$

Remark 1. The coefficients $s_{m,j}$ in (1.1) and the coefficients of the polynomials $A_{m,1}$ are well-known combinatorial numbers.

(a) $s_{m,j} = (-1)^{m-j} S(m+1, j+1)$ where $S(p, q)$ is the Stirling number of the second kind that counts the number of partitions of a p -set into q blocks. This follows immediately from the classical identity $k^{m+1} = \sum_{j=1}^{m+1} (-1)^{m+1-j} S(m+1, j) (k)_j$ (for example, see Stanley [14, 4.3,c]).

(b) For $m = 0$ we have $A_{0,1} = 1$ and $A_{m,1} = \sum_{j=1}^m A(m, j)t^j$ for $m > 0$ where $A(m, j)$ is the Eulerian number [14, 4.3,d].

1.2. **The monoid \mathbb{Z}_+^d .** Next we consider $M = \mathbb{Z}_+^d$. The crucial observation is that the problem is multiplicative for products of polynomials in disjoint variables. Suppose that $f(x) = g(y)h(z)$, $y = (x_1, \dots, x_r)$, $z = (x_{r+1}, \dots, x_d)$. Then

$$(1.5) \quad E_{M,f}(t) = \sum_{x \in \mathbb{Z}_+^d} f(x)t^{\deg x} = \left(\sum_{y \in \mathbb{Z}_+^r} g(y)t^{\deg y} \right) \left(\sum_{z \in \mathbb{Z}_+^{d-r}} h(z)t^{\deg z} \right)$$

by multiplication of power series.

In order to exploit (1.5) we split the last variable off,

$$f(x) = \sum_i f_i(x_1, \dots, x_{d-1})x_d^i,$$

and obtain

$$(1.6) \quad \begin{aligned} E_{M,f}(t) &= \sum_i \left(\left(\sum_{x' \in \mathbb{Z}_+^{d-1}} f_i(x')t^{\deg x'} \right) \left(\sum_{k=1}^{\infty} k^i t^{ki} \right) \right) \\ &= \sum_i \left(\frac{A_{i,u}(t)}{(1-t^u)^{i+1}} \sum_{x' \in \mathbb{Z}_+^{d-1}} f_i(x')t^{\deg x'} \right) \end{aligned}$$

with $u = \deg e_d$.

Applying this formula inductively allows us to eliminate all variables x_i and to end with the desired representation of $E_{\mathbb{Z}_+^d, f}(t)$.

Generalizing (1.4), let us consider the case in which f is a monomial, $f(x_1, \dots, x_d) = x_1^{m_1} \cdots x_d^{m_d}$, and \mathbb{Z}_+ is endowed with its *standard degree*, $\deg(x) = x_1 + \cdots + x_d$. Then equations (1.5) and (1.4) imply that

$$(1.7) \quad E_{M,f}(t) = \frac{m_1! \cdots m_d!}{(1-t)^{m_1 + \cdots + m_d + d}} + \text{terms of smaller pole order at } t = 1.$$

1.3. **Using the Stanley decomposition.** We now turn to general M . Normaliz computes a triangulation Σ of cone(M) into simplicial subcones σ . Moreover, it computes a *disjoint* decomposition

$$\text{cone}(M) = \bigcup_{\sigma \in \Sigma} \sigma \setminus S_\sigma$$

where S_σ is a union of facets of σ . The existence of such a decomposition is a nontrivial fact. Classically it is derived from the Brugesser-Mani theorem on the existence of line shellings (see Stanley [13]). Instead of a line shelling Normaliz (now) uses a method of Köppe and Verdoolaege: see [10] and [7, Section 4].

Every simplicial subcone (of full dimension) is generated by linearly independent vectors $v_1, \dots, v_d \in M$, $d = \text{rank } M$. They generate a free submonoid M_σ of M . For every σ Normaliz computes the set

$$E_\sigma = \{x \in \text{gp}(M) : x = \alpha_1 v_1 + \cdots + \alpha_d v_d, \alpha_i \in [0, 1)\}.$$

For $x \in E_\sigma$ we let $\varepsilon(x)$ be the sum of those v_i for which (i) $\alpha_i = 0$ and (ii) the facet of σ opposite to v_i lies in the excluded set S_σ (so that x lies in the excluded set). Then it is not hard to see that we have a disjoint decomposition

$$M = \bigcup_{\sigma \in \Sigma} \bigcup_{x \in E_\sigma} x + \varepsilon(x) + M_\sigma.$$

It is called a *Stanley decomposition* since its existence is originally due to Stanley [13].

In the following we set $\tilde{x} = x + \varepsilon(x)$ and

$$N_{\sigma,x} = \tilde{x} + M_\sigma.$$

Then

$$E_{M,f}(t) = \sum_{\sigma} \sum_{x \in E_\sigma} E_{N_{\sigma,x},f}(t).$$

Set $r = \text{rank } M$, and for given σ consider the linear map

$$\alpha_\sigma : \mathbb{Z}_+^d \rightarrow \mathbb{Z}^n, \quad \alpha_\sigma(y_1, \dots, y_d) = y_1 v_1 + \dots + y_d v_d,$$

where v_1, \dots, v_d is the generating set of M_σ as above. With

$$(1.8) \quad \begin{aligned} \deg_\sigma y &= \deg \alpha_\sigma(y), \\ g_{\sigma,x}(y) &= f(\alpha_\sigma(y) + \tilde{x}), \end{aligned}$$

we have

$$E_{N_{\sigma,x},f}(t) = t^{\deg \tilde{x}} \sum_{y \in \mathbb{Z}_+^d} g_{\sigma,x}(y) t^{\deg_\sigma y}.$$

This equation transforms the summation over $N_{\sigma,x}$ into a summation over \mathbb{Z}_+^d . Then we can apply (1.6) inductively to

$$(1.9) \quad \tilde{E}_{\sigma,f}(t) = \sum_{x \in E_\sigma} E_{N_{\sigma,x},f}(t).$$

Finally, we sum the rational functions $\tilde{E}_{\sigma,f}(t)$ over the triangulation Σ .

Remark 2. (a) Instead of applying (1.6) to every σ , we accumulate the polynomials $g_{\sigma,x}$ over all σ that induce the same degree \deg_σ on \mathbb{Z}_+^d (the classes formed in this way are called *denominator classes*).

(b) The time critical steps in the algorithm are

- (1) the coordinate transformation (1.8), and
- (2) the inductive application of (1.6).

In order to speed up (1), we factor the polynomial f , transform the factors separately, and multiply the transformed factors. If f happens to decompose into linear factors, then multiplication of linear polynomials becomes a time critical step. In order to speed up (2) we have introduced the denominator classes.

(c) Note that $\sum_{y \in \mathbb{Z}_+^d} g_{\sigma,x}(y) t^{\deg_\sigma y}$ is invariant under permutations of variables y_i that preserve the degrees $\deg_\sigma e_i$. Therefore one can go over $g_{\sigma,x}$ monomial by monomial and reorder the exponent vectors in such a way that the exponents of variables corresponding to the same degree become decreasing. The reordering significantly reduces the number

of monomials in the polynomials to which (1.6) must be applied, saves memory and also speeds up (1.6).

(d) We want to point out that (1.6) is *not* applied recursively. Instead the right hand side is expanded after the elimination of x_d , and x_{d-1} is then eliminated from the resulting polynomial whose coefficients are rational functions in t . This procedure is repeated until all x_i have been eliminated.

2. THE QUASIPOLYNOMIAL, ITS VIRTUAL LEADING COEFFICIENT, AND INTEGRATION

2.1. The quasipolynomial. All rational functions in t that come up in (1.9) can be written over the denominator

$$(1 - t^\ell)^{\deg f + \text{rank } M}$$

where ℓ is the least common multiple of the numbers $\deg x$ for the generators x of M that appear in the triangulation. This follows from (1.6) if one observes that $1 - t^\ell$ divides $1 - t^u$. Moreover, all summands have negative degree as rational functions in t . Therefore [14, 4.4.1] implies the following proposition.

Proposition 3.

$$E_{M,f}(t) = \sum_{k=0}^{\infty} q(k)t^k$$

where q is a rational quasipolynomial of period π dividing ℓ and of degree $\leq \deg f + \text{rank } M - 1$.

The statement about the quasipolynomial means that there exist polynomials $q^{(j)}$, $j = 0, \dots, \pi - 1$, of degree $\leq \deg f + \text{rank } M - 1$ such that

$$q(k) = q^{(j)}(k), \quad j \equiv k \pmod{\pi},$$

and

$$q^{(j)}(k) = q_0^{(j)} + q_1^{(j)}k + \dots + q_{\deg f + \text{rank } M - 1}^{(j)}k^{\deg f + \text{rank } M - 1}$$

with coefficients $q_i^{(j)} \in \mathbb{Q}$. As we will see below, it is justified to call

$$\text{ed}(M, f) = \deg f + \text{rank } M - 1$$

the *expected degree* of q .

2.2. The virtual leading coefficient and Lesbesgue integration. Let $m = \deg f$ and write $f = f_m + g$ where f_m is the degree m homogeneous component of m . Then $\deg g < m$, and it follows from Proposition 3 that g does not contribute to the coefficient $q_{\text{ed}(M,f)}^{(j)}$. Moreover, this coefficient is independent of j and given by an integral, as we will see in Proposition 4 below.

For the representation as an integral we must norm the measure in such a way that it is compatible with the lattice structure. We will integrate over the polytope

$$P = \text{cone}(M) \cap A_1, \quad A_1 = \{x \in \mathbb{R}^n : \deg x = 1\}.$$

Let $L_0 = L \cap \mathbb{R}M \cap A_0$ where $A_0 = \{x \in \mathbb{R}^n : \deg x = 0\}$ is the linear subspace of degree 0 elements. Then L_0 is a (saturated) sublattice of L of rank $d - 1$ ($d = \text{rank } M$), and

we choose a basis u_1, \dots, u_{d-1} of L_0 . Note that $H = \mathbb{R}M \cap A_1$ has dimension $d - 1$ and contains a point $z \in L$ since we have required that \deg takes the value 1 on $\text{gp}(M)$, and we can consider the *basic L_0 -simplex* $\delta = \text{conv}(z, z + u_1, \dots, z + u_{d-1})$ in H . Now we norm the Lesbesgue measure λ on H by giving volume $1/(d - 1)!$ to the basic L_0 -simplex. (The measure is independent of the choice of δ since two basic L_0 -simplices differ by an affine-integral automorphism of H .) We call λ the *L -Lesbesgue measure* on H .

Proposition 4. *For all $j = 0, \dots, \pi - 1$ one has*

$$(2.1) \quad q_{\text{ed}(M,f)}^{(j)} = \int_P f_m d\lambda.$$

Proof. We may assume that f is homogeneous of degree m . Let

$$L_c = \frac{1}{c}L.$$

Then

$$\int_P f_m d\lambda = \lim_{c \rightarrow \infty} \sum_{x \in P \cap L_c} \frac{1}{c^{d-1}} f(x)$$

by elementary integration theory.

Note that

$$f(x) = \frac{1}{c^m} f(cx)$$

by homogeneity and that $x \in P \cap L_c$ if and only $cx \in L \cap cP$. Thus

$$\int_P f_m d\lambda = \lim_{c \rightarrow \infty} \sum_{y \in cP \cap L} \frac{1}{c^{m+d-1}} f(y).$$

On the other hand, we obtain $q_{\text{ed}(M,f)}^{(j)}$ as the limit over the subsequence $(b\pi + j)_{b \in \mathbb{Z}_+}$:

$$q_{\text{ed}(M,f)}^{(j)} = \lim_{b \rightarrow \infty} \sum_{y \in (b\pi + j)P \cap L} \frac{1}{(b\pi + j)^{m+d-1}} f(y)$$

by Proposition 3. This concludes the proof. \square

In view of Proposition 4 it is justified to call $q_{\text{ed}(M,f)} = q_{\text{ed}(M,f)}^{(j)}$ the *virtual leading coefficient*, and the proposition justifies the term “expected degree” for $\deg f + \text{rank } M - 1$ the. In analogy with the definition of multiplicity in commutative algebra (for example, see [5]), we call

$$\text{vmult}(M, f) = \text{ed}(M, f)! q_{\text{ed}(M,f)}$$

the *virtual multiplicity of (M, f)* . It is an integer if P is a lattice polytope and f_m has integral coefficients, as we will see below.

2.3. Computing the integral. It is natural to compute the integral by summation over the triangulation: the triangulation of $\text{cone}(M)$ into simplicial subcones σ induces a triangulation of the polytope P into simplices $\delta = \sigma \cap P$. As usual let $v_1, \dots, v_d \in M$ be the generators of σ . Then δ is spanned by the degree 1 vectors $v_i / \deg(v_i)$, $i = 1, \dots, d$. Let e_1, \dots, e_d be the unit vectors in \mathbb{R}^d . Then the substitution $e_i \mapsto v_i / \deg(v_i)$ induces a linear map $\mathbb{R}^d \rightarrow \mathbb{R}M$ that in its turn restricts to an affine map α from the standard degree 1 hyperplane in \mathbb{R}^d spanned by e_1, \dots, e_d to the hyperplane $A_1 \cap \mathbb{R}M$, and the image of the unit simplex Δ is just δ .

Proposition 5. *One has*

$$(2.2) \quad \int_{\delta} f d\lambda = \frac{|\det(v_1, \dots, v_d)|}{\deg(v_1) \cdots \deg(v_d)} \int_{\Delta} (f \circ \alpha) d\mu$$

where μ is the \mathbb{Z}^d -Lebesgue measure on the hyperplane of standard degree 1 in \mathbb{R}^d .

Proof. This is just the substitution rule if one observes that the absolute value of the functional determinant of $\alpha|_H$ is given by the factor in front of the integral. For an affine map the functional determinant is constant. So we can assume $f = 1$ and it remains to relate the volumes of δ and Δ . But Δ has volume $1/(d-1)!$ with respect to μ and δ has volume

$$\frac{1}{(d-1)!} \frac{|\det(v_1, \dots, v_d)|}{\deg(v_1) \cdots \deg(v_d)};$$

with respect to λ ; see [7, Section 4]. □

After the substitution it remains to evaluate the integral over Δ , and this can be done monomial by monomial:

Proposition 6.

$$(2.3) \quad \int_{\Delta} y_1^{m_1} \cdots y_d^{m_d} d\mu = \frac{m_1! \cdots m_d!}{(m_1 + \cdots + m_d + d - 1)!}$$

Proof. Let $g = y_1^{m_1} \cdots y_d^{m_d}$ and $M = \mathbb{Z}_d^+$. Then

$$E_{M,g}(t) = \frac{m_1! \cdots m_d!}{(1-t)^{(m_1 + \cdots + m_d + d)}} + \text{terms of smaller pole order at } t = 1,$$

as stated in (1.7).

The quasipolynomial is a true polynomial in this case, and the (virtual) multiplicity is given by the value of the numerator polynomial at $t = 1$, namely $m_1! \cdots m_d!$ (for example, see [5, 4.1.9]). Now Proposition 4 gives the integral. □

3. COMPUTATIONAL EXAMPLES

We illustrate the use of `NmzIntegrate` by three related examples coming from combinatorial voting theory that are discussed in [12]. We refer the reader to [11], [12] or [15] for a more extensive treatment.

Consider an election in which each of the k voters fixes a linear preference order of n candidates. In other words, voter i chooses a linear order of the candidates $1, \dots, n$. Each such order represents a permutation of $1, \dots, n$. Set $N = n!$. The result of the election is an

N -tuple (x_1, \dots, x_N) in which x_p is the number of voters that have chosen the preference order labeled p . Then $x_1 + \dots + x_N = k$, and (x_1, \dots, x_N) can be considered as a lattice point in the positive orthant of \mathbb{R}_+^N , or, more precisely, as a lattice point in the simplex

$$U_k^{(n)} = \mathbb{R}_+^N \cap A_k = k(\mathbb{R}_+^N \cap A_1) = kU^{(n)}$$

where A_k is the hyperplane defined by $x_1 + \dots + x_N = k$, and $U^{(n)} = U_1^{(n)}$ is the unit simplex of dimension $N - 1$ naturally embedded in N -space. We assume that all lattice points in the simplex $U_k^{(n)}$ have equal probability of being the outcome of the election.

The following three problems have been considered in [12] for 4 candidates A, B, C, D :

- (1) the Condorcet paradox,
- (2) the Condorcet efficiency of plurality voting,
- (3) plurality voting versus cutoff.

For $n = 4$ one has $N = 24$, and the dimension of the polytope $U^{(4)}$ is already quite large.

Let us say that candidate A *beats* candidate B if the number of voters that prefer candidate A to candidate B is larger than the number of voters with the opposite preference. Candidate A is the *Condorcet winner* if A beats all other candidates. As the Marquis de Condorcet noticed, the relation “beats” is nontransitive for some outcomes of the election, and there may be no Condorcet winner. This phenomenon is called the *Condorcet paradox*. Problem (1) asks for its asymptotic probability as the number k of voters goes to ∞ , or even for the precise number of election results without a Condorcet winner, depending on the number k of voters.

It is not hard to see that the outcomes that have A as the Condorcet winner can be described by three homogeneous linear inequalities whose coefficients are given in Table 1 (relative to the lexicographic order of the permutations of A, B, C, D). They cut out a

λ_1 :	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	-1
λ_2 :	1	1	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1
λ_3 :	1	1	1	1	1	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

TABLE 1. Inequalities expressing that A beats the other 3 candidates

rational polytope from $U^{(n)}$, and the probability of Condorcet’s paradox can be computed from the volume of the polytope. Finding the precise number of election results without (or with) a Condorcet winner requires the computation of the Ehrhart function of the semi-open polytope. Neither Normaliz nor NmzIntegrate can yet compute Ehrhart series for semi-open polytopes directly, but it is always possible to fall back on inclusion/exclusion.

We refer the reader to [7] for a description of problems (2) and (3) and for the systems of linear inequalities to be solved in each case. Normaliz 2.8 can indeed compute the volumes and the Ehrhart series in dimension 24 that arise from tasks (1), (2) and (3) despite the fact that the triangulations to be evaluated for (2) and (3) are formidable (see Table 3 or [7]).

As Schürmann [12] observed, the computations can be considerably simplified by exploiting the symmetries in the inequalities: some variables share the same coefficients in each inequality, for example the first 6 variables in Table 1. Therefore they can be

replaced by their sum, and the replacement constitutes a projection of the original polytopes, monoids or cones onto objects of smaller dimension. For the Condorcet paradox the system of inequalities reduces to Table 2. However, instead of simply counting lattice

$$\begin{array}{cccccccc} 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \end{array}$$

TABLE 2. Inequalities exploiting the symmetries in Table 1

points, one must now count them with their numbers of preimages. These are given by polynomials, namely products of binomial coefficients. In our example the polynomial is

$$\binom{y_1+5}{5}(y_2+1)(y_3+1)(y_4+1)(y_5+1)(y_6+1)(y_7+1)\binom{y_8+5}{5}$$

where $y_1 = x_1 + \dots + x_6$ etc. In other words, the Ehrhart function (or the volume) of a high dimensional polytope is replaced by a generalized Ehrhart function of a polytope of much lower dimension (or the virtual leading coefficient of the quasipolynomial).

A priori it may not be clear that the replacement of combinatorial complexity in high dimension by multivariate polynomial arithmetic in low dimension pays dividends, but this is indeed the case. Tables 3 and 4 compare both approaches for the Ehrhart series. The computations were run on a SUN xFire 4450 with 20 parallel threads (but the number of active threads is limited by the size of the triangulation). Volume computations are faster by a factor of approximately 3 in each case.

computation	triangulation size	real time
Condorcet paradox	1,473,107	00:00:30 h
Condorcet efficiency	347,225,775,338	218:13:55 h
plurality vs. cutoff	257,744,341,008	175:11:26 h

TABLE 3. Computation times for original data

computation	rank	degree of polynomial	triangulation size	real time
Condorcet paradox	8	16	17	0.5 sec
Condorcet efficiency	13	11	17,953	5:49:29 h
plurality vs. cutoff	6	18	3	32.5 sec

TABLE 4. Computation times for symmetrized data

A welcome side effect of the computations of the generalized Ehrhart functions is that they have confirmed the results reported on in [7].

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