“Chordal” package: Exploiting graphical structure in polynomial ideals

Diego Cifuentes

Laboratory for Information and Decision Systems
Electrical Engineering and Computer Science
Massachusetts Institute of Technology

Joint work with **Pablo A. Parrilo** (MIT)
`arXiv:1411.1745, 1507.03046, 1604.02618`

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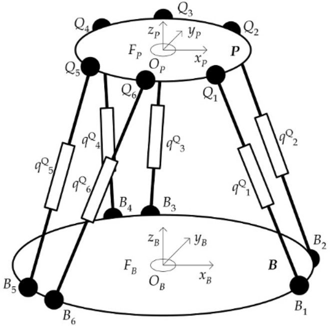
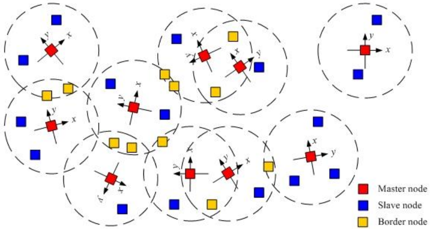
Polynomial systems

Systems of polynomial equations have been used to model problems in areas such as: robotics, cryptography, statistics, optimization, computer vision, power networks, graph theory.





$$S + E \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} ES \underset{k_{-2}}{\overset{k_{+2}}{\rightleftharpoons}} P + E$$

$$\frac{d[S]}{dt} = -k_1[E][S] + k_{-1}[ES]$$

$$\frac{d[E]}{dt} = -k_1[E][S] + (k_{-1} + k_2)[ES] - k_{-2}[E][P]$$

$$\frac{d[ES]}{dt} = k_1[E][S] - (k_{-1} + k_2)[ES] + k_{-2}[E][P]$$

Polynomial systems

Systems of polynomial equations have been used to model problems in areas such as: robotics, cryptography, statistics, optimization, computer vision, power networks, graph theory.

Given polynomial equations $F = \{f_1, \ldots, f_m\}$, let

$$\mathcal{V}(F) := \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0\}$$

denote the associated variety.

Depending on the application we might be interesting in:

- **Feasibility** Is there any solution, i.e., $\mathcal{V}(F) \neq \emptyset$?
- **Counting** How many solutions?
- **Dimension** What is the dimension of $\mathcal{V}(F)$?
- **Components** Decompose $\mathcal{V}(F)$ into irreducible components.

Polynomial systems and graphs

Systems coming from applications often have simple *sparsity structure*.
We can represent this structure using graphs.

Given m equations in n variables, construct a graph as:

- Nodes are the variables $\{x_0, \ldots, x_{n-1}\}$.
- For each equation, add a clique connecting the variables appearing in that equation

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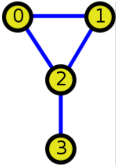
Example:

$$F = \{x_0^2 x_1 x_2 + 2x_1 + 1,\ \ x_1^2 + x_2,\ \ x_1 + x_2,\ \ x_2 x_3\}$$



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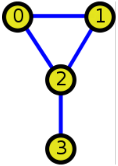

$$F = \{x_0^2 x_1 x_2 + 2x_1 + 1, \;\; x_1^2 + x_2, \;\; x_1 + x_2, \;\; x_2 x_3\}$$

Question: Can the graph structure help *solve* polynomial systems?

Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, ...

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, ...

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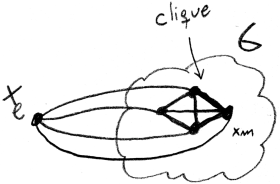
We hope to change this... ;)

Chordal graphs

For a graph G, an ordering of its vertices $x_0 > x_1 > \cdots > x_{n-1}$ is a *perfect elimination ordering* if for each x_ℓ

$$X_\ell := \{x_m : x_m \text{ is adjacent to } x_\ell,\ \ x_\ell > x_m\}$$

is a clique.
A graph is chordal if it has a perfect elimination ordering.



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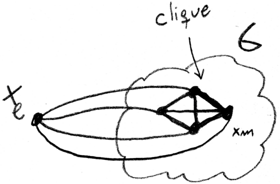

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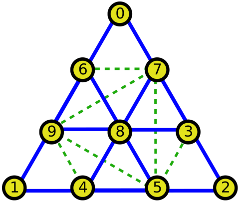
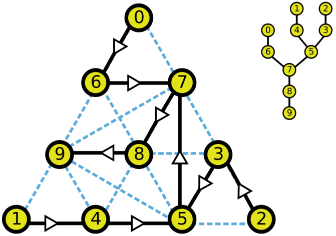
A *chordal completion* of G is supergraph that is chordal.



Elimination tree of a chordal graph

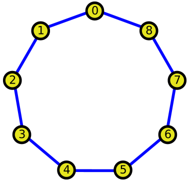
The elimination tree of a graph G is the following *directed spanning tree*:

For each ℓ there is an arc towards its smallest neighbor p, with $p > \ell$.



Example 1: Coloring a cycle

Let $C_n = (V, E)$ be the cycle graph and consider the ideal I given by the equations



$$x_i^3 - 1 = 0, \qquad i \in V$$
$$x_i^2 + x_i x_j + x_j^2 = 0, \qquad ij \in E$$

These equations encode the proper 3-colorings of the graph. Note that coloring the cycle graph is very easy!

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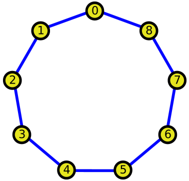

$$\begin{aligned} x_i^3 - 1 = 0, \qquad & i \in V \\ x_i^2 + x_i x_j + x_j^2 = 0, \qquad & ij \in E \end{aligned}$$

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However, a Gröbner basis is not so simple: one of its 13 elements is

$$x_0x_2x_4x_6 + x_0x_2x_4x_7 + x_0x_2x_4x_8 + x_0x_2x_5x_6 + x_0x_2x_5x_7 + x_0x_2x_5x_8 + x_0x_2x_6x_8 + x_0x_2x_7x_8 + x_0x_2x_8^2 + x_0x_3x_4x_6 + x_0x_3x_4x_7$$
$$+x_0x_3x_4x_8 + x_0x_3x_5x_6 + x_0x_3x_5x_7 + x_0x_3x_5x_8 + x_0x_3x_6x_8 + x_0x_3x_7x_8 + x_0x_3x_8^2 + x_0x_4x_6x_8 + x_0x_4x_7x_8 + x_0x_4x_8^2 + x_0x_5x_6x_8$$
$$+x_0x_5x_7x_8 + x_0x_5x_8^2 + x_0x_6x_8^2 + x_0x_7x_8^2 + x_0 + x_1x_2x_4x_6 + x_1x_2x_4x_7 + x_1x_2x_4x_8 + x_1x_2x_5x_6 + x_1x_2x_5x_7 + x_1x_2x_5x_8$$
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Elimination ideals

The elimination ideals of an ideal $I \subset \mathbb{K}[x_0, \cdots x_{n-1}]$ are

$$\begin{aligned} I_0 &:= I \\ I_1 &:= I \cap \mathbb{K}[x_1, x_2, x_3, \cdots x_{n-1}] \\ I_2 &:= I \cap \mathbb{K}[x_2, x_3, \cdots x_{n-1}] \\ I_3 &:= I \cap \mathbb{K}[x_3, \cdots x_{n-1}] \\ &\vdots \\ I_{n-1} &:= I \cap \mathbb{K}[x_{n-1}] \end{aligned}$$

The system of equations is feasible if and only if $I_{n-1} \neq \langle 1 \rangle$. We can also find a solution by backtracking.

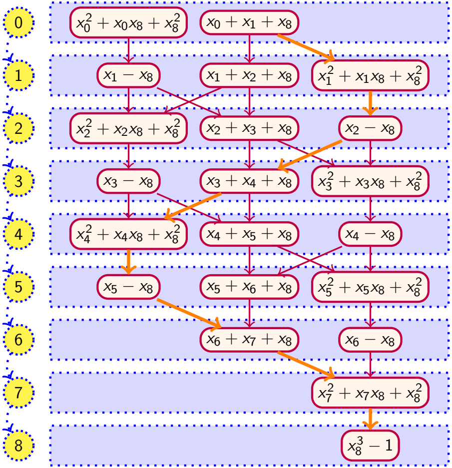
Example 1: Coloring a cycle

There is an alternative representation of the ideal, that respects its graphical structure.

The variety can be decomposed into *triangular* sets:

$$\mathcal{V}(I) = \bigcup_{T} \mathcal{V}(T)$$

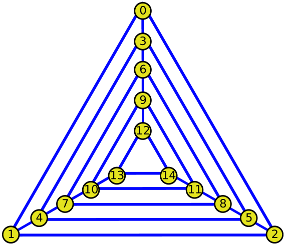
where the union is overall all *maximal directed paths* (or *chains*). The number of triangular sets is 21, which is the 8-th Fibonacci number.



Example 2: Minimal vertex covers

Let $G = C_3 \times P_n$ be a graph of nested triangles. Consider the minimal vetex cover problem.

Find a minimal subset of $S \subset V$ such that every edge is incident to at least one vertex in S.



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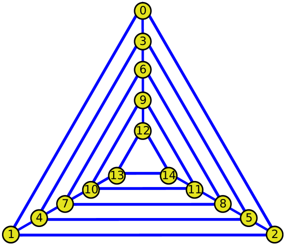


We can solve this problem algebraically using the *edge ideal*

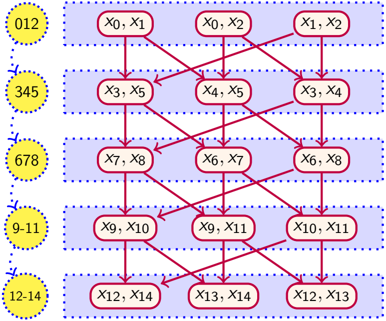
$$I(G) := \langle x_i x_j : ij \in E \rangle$$

The minimal vertex covers of G are in bijection with the irreducible components of $I(G)$.

Example 2: Minimal vertex covers

For the graph of nested triangles, ideal $I(G)$ has $3 \times 2^{n-1}$ components.

They correspond to the maximal directed paths in the diagram.



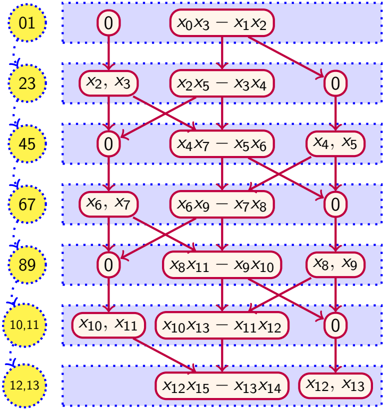
Example 3: Ideal of adjacent minors

$$I = \{x_{2i}x_{2i+3} - x_{2i+1}x_{2i+2} : 0 \leq i < n\}$$

This is the ideal of adjacent minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & x_4 & \cdots & x_{2n-2} \\ x_1 & x_3 & x_5 & \cdots & x_{2n-1} \end{pmatrix}$$

The total number of irreducible components is the n-th Fibonacci number.



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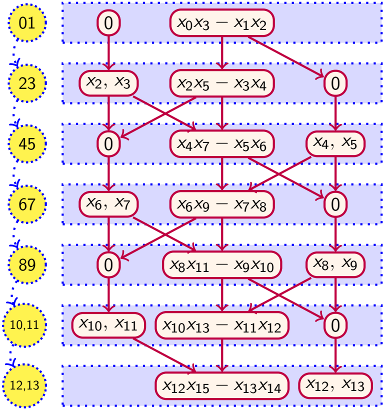


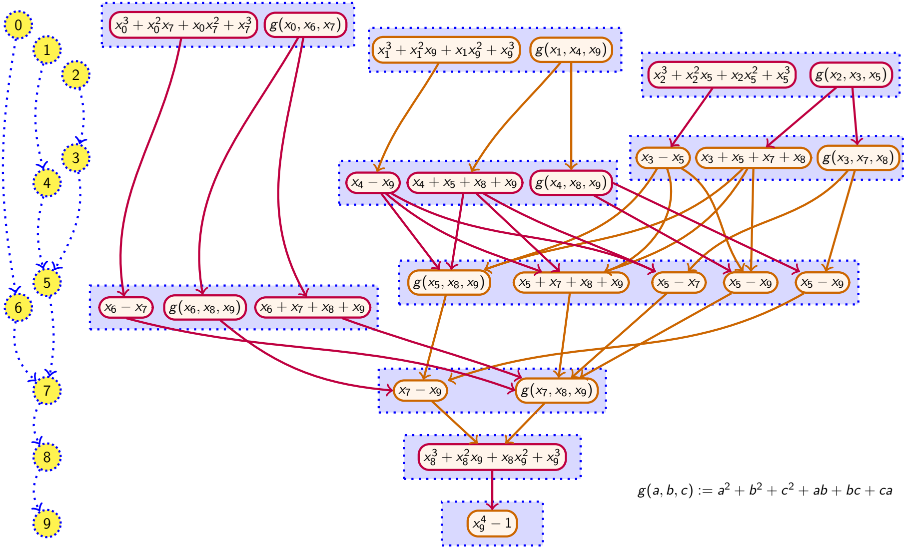
More generally, the ideal of adjacent minors of a $k \times n$ matrix also has a simple chordal network representation.

Chordal networks

A *G-chordal network* is a directed graph $\mathcal{N}$, whose nodes are polynomial sets, satisfying the following conditions

- arcs follow elimination tree: if (F_ℓ, F_p) is an arc, then (ℓ, p) is an arc of the elimination tree, where $\ell = \text{rank}(F_\ell), p = \text{rank}(F_p)$.
- nodes supported on cliques: each node F of $\mathcal{N}$ is given a rank $\ell := \text{rank}(F)$, such that F only involves variables in the clique X_ℓ.

Chordal networks (Example)



$$g(a, b, c) := a^2 + b^2 + c^2 + ab + bc + ca$$

Computing chordal networks: Triangular sets

Defn: A zero dimensional triangular set is $T = \{t_0, \ldots, t_{n-1}\}$ such that

$$
\begin{aligned}
t_0 &= x_0^{d_0} + g_0(x_0, x_1, \ldots, x_{n-1}), && (\deg_{x_0}(g_0) < d_0)\\
&\vdots \\
t_{n-2} &= x_{n-2}^{d_{n-2}} + g_{n-2}(x_{n-2}, x_{n-1}), && (\deg_{x_{n-2}}(g_1) < d_{n-2})\\
t_{n-1} &= g_{n-1}(x_{n-1})
\end{aligned}
$$

Remk: A triangular set is a Gröbner basis w.r.t. lexicographic order.

Defn: Let $I \subset \mathbb{K}[X]$ be a zero dimensional ideal. A triangular decomposition of I is a collection $\mathcal{T}$ of triangular sets, such that

$$\mathcal{V}(I) = \bigsqcup_{T \in \mathcal{T}} \mathcal{V}(T)$$

Computing chordal networks (Example)

The ideal

$$I = \langle x_0x_2 - x_2, x_0^3 - x_0, x_1 - x_2, x_2^2 - x_2, x_2 - x_3 \rangle$$

can be decomposed into three triangular sets

$$T_1 = (x_0^3 - x_0,\;\; x_1 - x_2,\;\; x_2,\;\; x_3),$$
$$T_2 = (x_0 - 1,\;\; x_1 - x_2,\;\; x_2 - 1,\;\; x_3),$$
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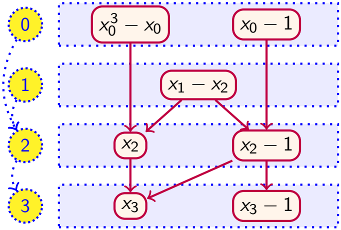
These triangular sets correspond to chains of a chordal network



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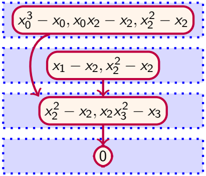

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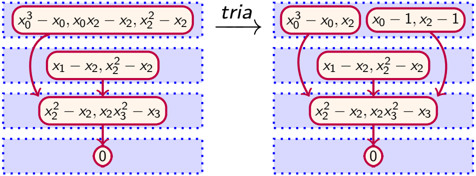

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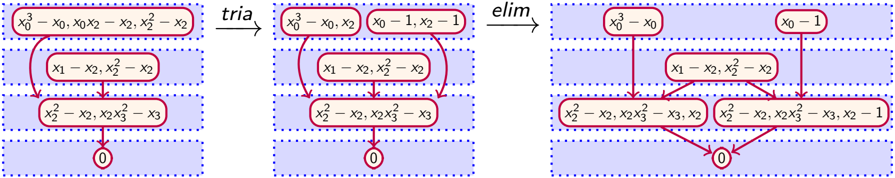

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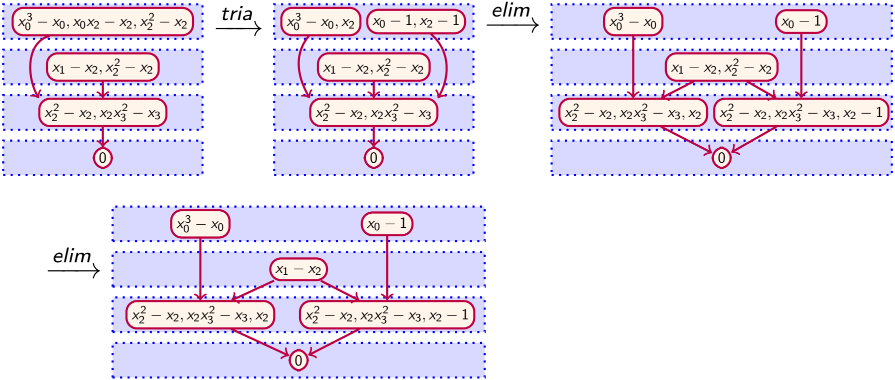

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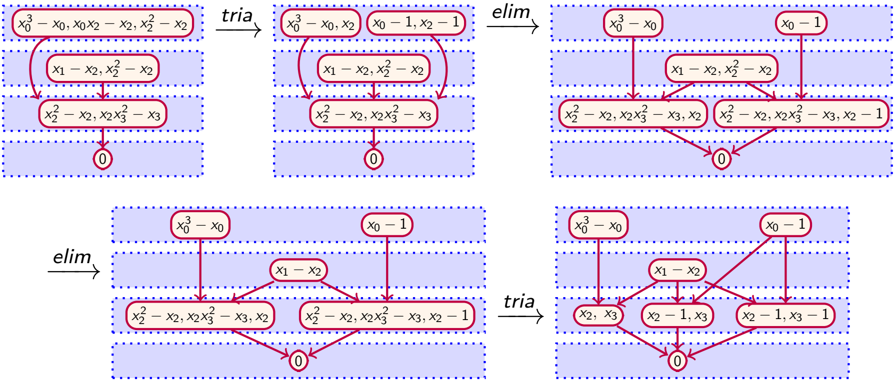

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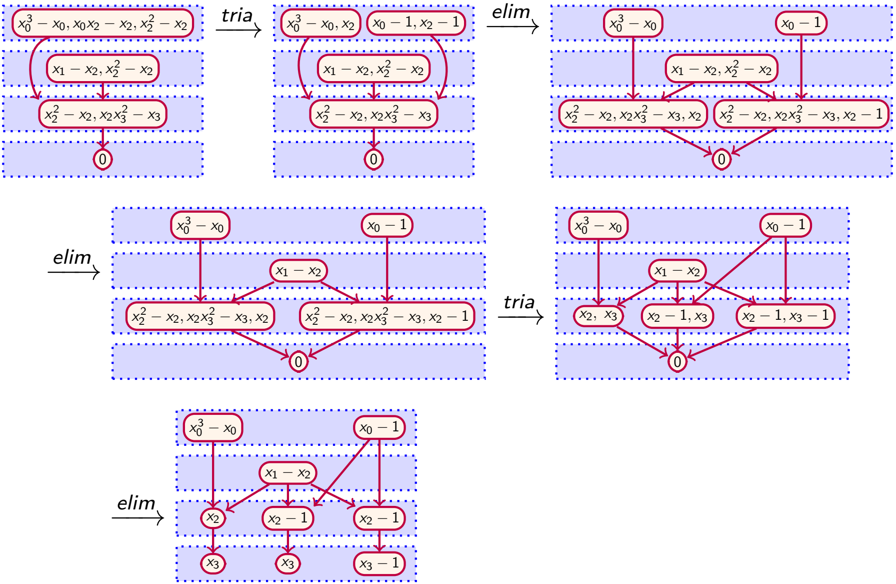

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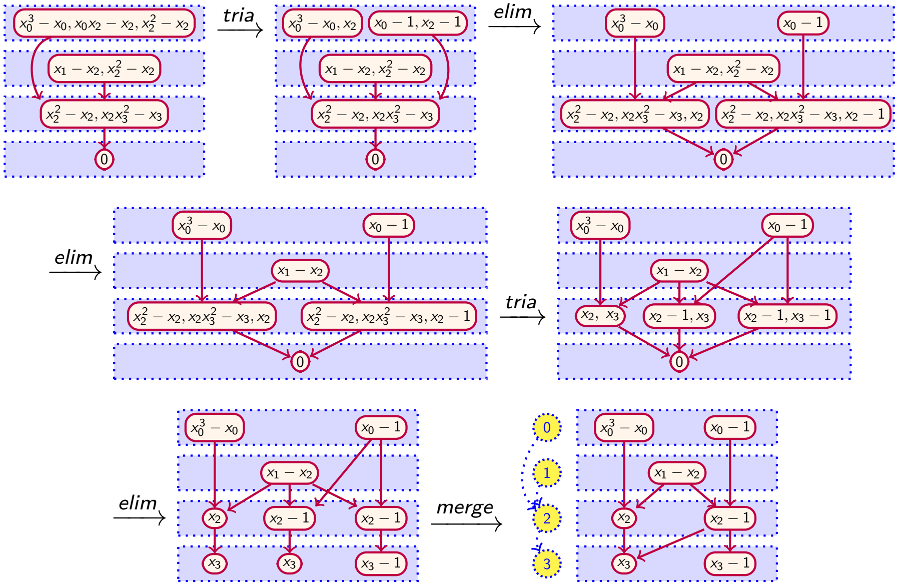

Main results

Thm 1: Chordal triangularization obtains a G-chordal network, whose chains give a triangular decomposition of F.

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For "nice" cases the chordal network obtained has linear size.

Thm 2: Let $\mathcal{F}$ be a family of structured polynomial systems such that $|\mathcal{V}(F \cap \mathbb{K}[X_l])|$ is bounded for any $F \in \mathcal{F}$ and for any maximal clique X_l. Then any $F \in \mathcal{F}$ admits a chordal network representation of size $O(n)$.

Chordal networks in computational algebra

Given a triangular chordal network $\mathcal{N}$ of an ideal I, we can compute in linear time:

- the cardinality of $\mathcal{V}(I)$.
- the dimension of $\mathcal{V}(I)$
- the top dimensional part of $\mathcal{V}(I)$.

We also show efficient algorithms for:

- radical ideal membership.
- computing equidimensional (sometimes irreducible) components.

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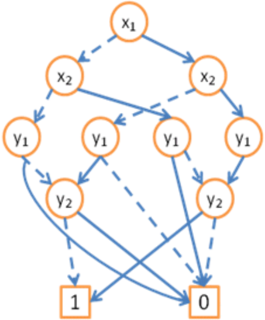
The main difficulty is that there might be exponentially many chains. It can be overcomed by cleverly using dynamic programming (or message-passing).

Links to BDDs

Very interesting connections with binary decision diagrams (BDDs).

- A clever representation of Boolean functions/sets, usually much more compact than naive alternatives
- Enabler of very significant practical advances in (discrete) formal verification and model checking
- "One of the only really fundamental data structures that came out in the last twenty-five years" (D. Knuth)

For the special case of monomial ideals, chordal networks are equivalent to (reduced, ordered) BDDs. But in general, more powerful!



Summary

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- Many classes of ideals admit simple chordal network representations.
- Try our Macaulay2 package!!!

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If you want to know more:

- D. Cifuentes, P.A. Parrilo (2017), Chordal networks of polynomial ideals. *SIAM J. of Applied Algebra and Geometry*, 1(1):73-170. `arXiv:1604.02618`.
- D. Cifuentes, P.A. Parrilo (2016), Exploiting chordal structure in polynomial ideals: a Gröbner basis approach. *SIAM J. Discrete Math.*, 30(3):1534-1570. `arXiv:1411.1745`.
- D. Cifuentes, P.A. Parrilo (2016), An efficient tree decomposition method for permanents and mixed discriminants. *Linear Alg. and its Appl.*, 493:45-81. `arXiv:1507.03046`.

Thanks for your attention!