Introduction to Numerical Algebraic Geometry

What is possible in Macaulay2?

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Polynomial homotopy continuation

- Target system: n equations in n variables,

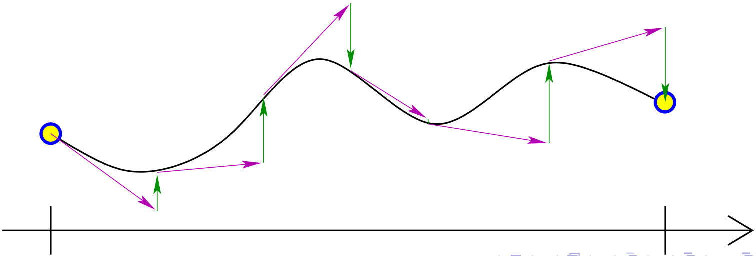
$$F(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})) = \mathbf{0},$$

  where $f_i \in R = \mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, ..., x_n]$ for $i = 1, ..., n$.
- Start system: n equations in n variables:

$$G(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x})) = \mathbf{0},$$

  such that it is easy to solve.
- Homotopy: for $\gamma \in \mathbb{C} \setminus \{0\}$ consider

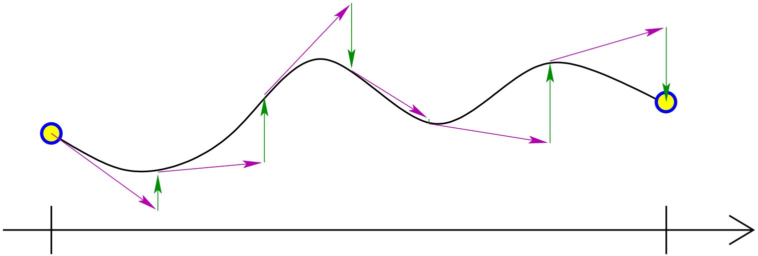
$$H(\boldsymbol{x}, t) = (1 - t)G(\boldsymbol{x}) + \gamma t F(\boldsymbol{x}), \ t \in [0, 1].$$



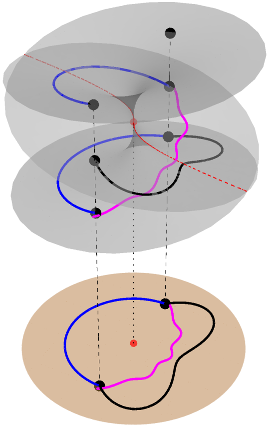
Numerical linear algebra under the hood: from polynomial systems to ODEs

target		start
$f_1 = x_1^4 x_2 + 5x_1^2 x_2^3 + x_1^3 - 4$		$g_1 = x_1^5 - 1$
$f_2 = x_1^2 - x_1 x_2 + x_2 - 8$		$g_2 = x_2^2 - 1$
{target solutions}	$\longleftarrow$	{start solutions}

$H(\boldsymbol{x}, t) = 0$ implies $\dfrac{d\boldsymbol{x}}{dt} = -\left(\dfrac{\partial H}{\partial \boldsymbol{x}}\right)^{-1} \dfrac{\partial H}{\partial t}$.



Continuation in a nutshell: lifting paths from B to V.



- The covering map, $\pi : V \to B$, from total space, the set of pairs (problem, solution),

$$V = \{(a, x) \in B \times \mathbb{C} \mid x^3 - a = 0\}$$

to base space, a parameterized space of problems,

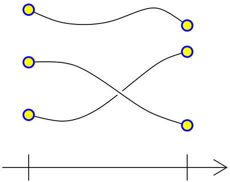
$$B = \{a \in \mathbb{C} \mid x^3 - a = 0 \text{ has 3 solutions}\}$$
$$= \mathbb{C} \setminus D, \text{ where } D = \{0\} \text{ is the branch locus.}$$

- Target system F and start system G are points in the base space B.
- Usually it is necessary to randomize a path in B to avoid the branch locus D.
- *All* ingredients in this picture are up to discussion. How to construct a suitable family? How to choose a start system?

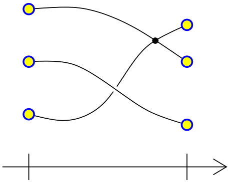
Global picture

Optimal homotopy:

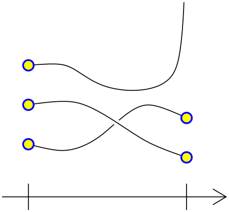
- the continuation paths are regular;
- the homotopy establishes a bijection between the start and target solutions.



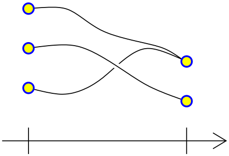
Possible singular scenarios:



non-generic



diverging paths



multiple solutions

Global picture

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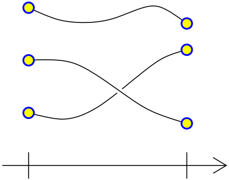

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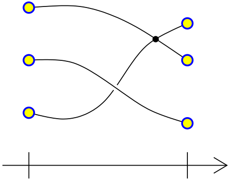

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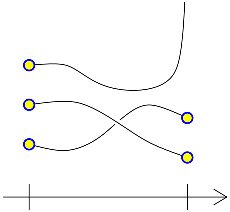

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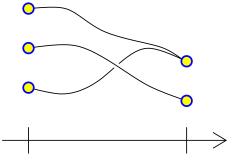
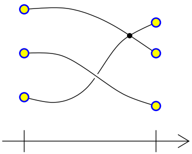


multiple solutions

- Randomization:



For all but finite number of $\gamma \in \mathbb{C}$ the homotopy

$$H(\boldsymbol{x}, t) = (1-t)G(\boldsymbol{x}) + \gamma t F(\boldsymbol{x}).$$

is regular for $0 \leq t < 1$.

- Book: Sommese and Wampler, The numerical solution of systems of polynomials (2005).
- Software:
  - PHCpack (Verschelde);
  - HOM4PS (group of T.Y.Li);
  - Bertini (group of Sommese);
  - NAG4M2: Numerical Algebraic Geometry for Macaulay2 (L.).
  - New framework: MonodromySolver (Duff, Hill, Jensen, Lee, L., and Sommars).

Poster by Kisun Lee next Monday (at reception)

Talk by Tim Duff next Friday (10:30-10:55, Skiles 255)

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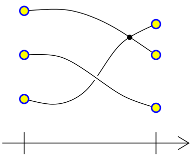

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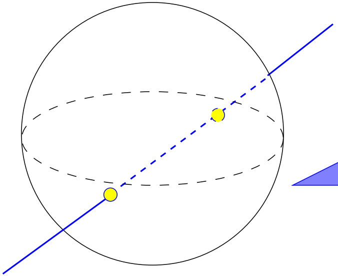
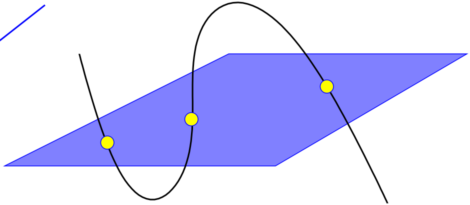
Higher-dimensional solution sets

- Let $I = (f_1, \ldots, f_N)$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$.
- Goal: Understand the variety

$$X = \mathbb{V}(I) = \{\boldsymbol{x} \in \mathbb{C}^n \mid \forall f \in I,\ f(\boldsymbol{x}) = 0\}.$$

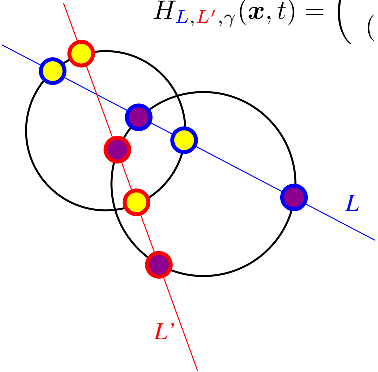
- A witness set for an equidimensional component Y of X:
  - a generic "slicing" plane L with $\dim L = \operatorname{codim} Y$
  - witness points $w_{Y,L} = Y \cap L$
  - (generators of I)

Numerical irreducible decomposition

- Homotopy mapping $w_{Y,L} \to w_{Y,L'}$:

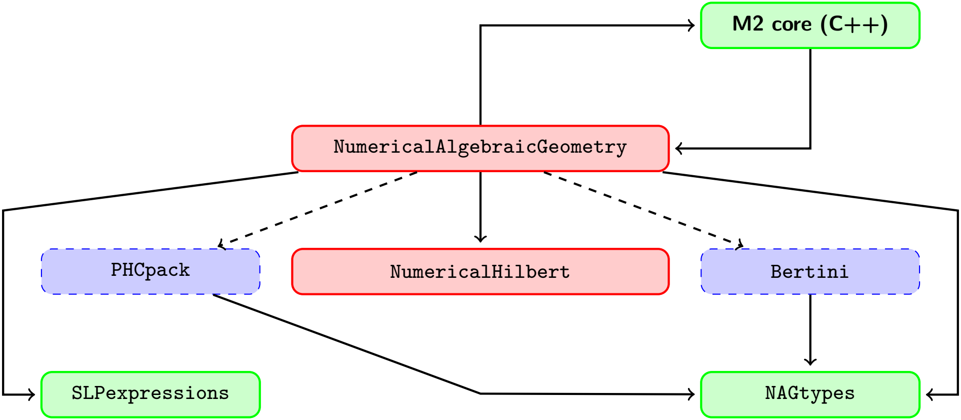
$$H_{L,L',\gamma}(\boldsymbol{x},t) = \begin{pmatrix} f(\boldsymbol{x}) \\ (1-t)L(\boldsymbol{x}) + \gamma t L'(\boldsymbol{x}) \end{pmatrix}, \quad t \in [0,1].$$



- Monodromy action: a permutation on $w_{Y,L}$ is produced by homotopy $H_{L,L',\gamma}$ followed by $H_{L',L,\gamma'}$ for random $\gamma, \gamma' \in \mathbb{C}$.
- Irreducible decomposition: a partition of the witness set $w_{Y,L}$ stable under this action.

Dependencies

Macaulay2 core implements its own fast homotopy continuation.
The interfaces are optional!



Related numerical packages

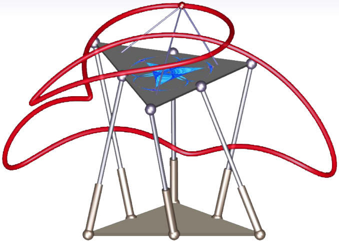
Classical numerical AG functionality:

- core dependencies: `NAGtypes` and `SLPexpressions`
- Macaulay dual spaces: `NumericalHilbert` (Krone)
- interfaces: `PHCpack` (Verschelde, Gross, Petrovic) and `Bertini` (Rodriguez, Gross, Bates, L.)

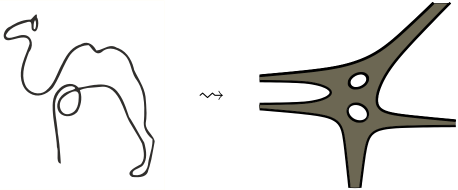
Applications and advanced functionality:

- `MonodromySolver` (Duff, Hill, Jensen, Lee, L., Sommars)
- `NumericalCertification` (Lee)
- `NumericalImplicitization` (Chen, Kileel)
- `NumericalSchubertCalculus` (L., Verschelde, del Campo, Sottile, Vakil)
- ...

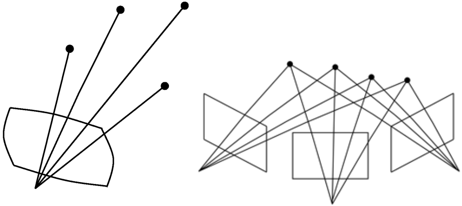
Applications



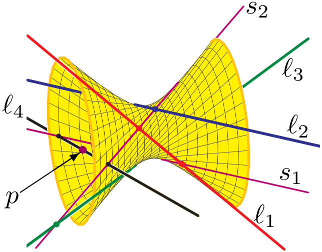
Kinematics: robot systems.



Tropical geometry: tropical curves



Computer vision: minimal problems



Enumerative AG: Schubert calculus